LINEAR ALGEBRA — PRACTICE EXAM 3

(1) **Matrices for linear transformations between general vector spaces.** Let $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

Let $\mathfrak{B} = \{2, 1+t, t^2\}$ be a basis for $\mathbb{P}_2$ and let

$$\mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. Compute the matrix for T with respect to these bases.

ANSWER: We compute B column by column:

$$\begin{aligned}
[T(2)]_{\mathfrak{D}} &= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}_{\mathfrak{D}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\
[T(1+t)]_{\mathfrak{D}} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathfrak{D}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\
[T(t^2)]_{\mathfrak{D}} &= \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}_{\mathfrak{D}} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix},
\end{aligned}$$

So $B = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

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(2) **Determinants.** Compute the determinant of

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}.$$

ANSWER: The determinant can be computed directly, but I will do some row operations first to make these computations easier.

$$\begin{aligned}
\det(A) &= \det \begin{bmatrix} 0 & 0 & 6 & -4 \\ 0 & 3 & 5 & -8 \\ 0 & -12 & 1 & 16 \\ 1 & -4 & 0 & 6 \end{bmatrix} \begin{cases} I - 2IV \\ II - 3IV \\ III + 3IV \end{cases} \\
&= \det \begin{bmatrix} 0 & 0 & 6 & -4 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 21 & -16 \\ 1 & -4 & 0 & 6 \end{bmatrix} \{ III - 4II
\end{aligned}$$

$$= \det \begin{bmatrix} 0 & 0 & 6 & -4 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & -3 & 0 \\ 1 & -4 & 0 & 6 \end{bmatrix} \{III - 4I$$

Now I compute the determinant:

$$\begin{aligned} &\det \begin{bmatrix} 0 & 0 & 6 & -4 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & -3 & 0 \\ 1 & -4 & 0 & 6 \end{bmatrix} \quad \text{(expand across the third row)} \\ &= -3 \det \begin{bmatrix} 0 & 0 & -4 \\ 0 & 3 & -8 \\ 1 & -4 & 6 \end{bmatrix} \quad \text{(expand down the first column)} \\ &= -3 \det \begin{bmatrix} 0 & -4 \\ 3 & -8 \end{bmatrix} \\ &= -3(12) = -36. \end{aligned}$$

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(3) **Characteristic polynomial and eigenvalues.** Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}.$$

Compute the characteristic polynomial of A, and find its eigenvalues.

ANSWER:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{bmatrix} \\ &= (5-\lambda) \det \begin{bmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{bmatrix} \\ &= (5-\lambda)((3-\lambda)(7-\lambda)+2) \\ &= (5-\lambda)(\lambda^2 - 10\lambda + 23). \end{aligned}$$

The quadratic term does not factor, so we can use the quadratic formula to find the three eigenvalues:

$$\lambda = 5, \frac{10 \pm \sqrt{100 - 4(23)}}{2}.$$

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(4) **Eigenvectors and eigenspaces.** Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

Find a basis for the eigenspaces corresponding to the eigenvalues $\lambda = 1, 2, 3$. Verify that your basis vectors are, indeed, eigenvectors for their eigenvalues.

ANSWER:

$$E_1 = \ker \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_2 = \ker \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} = \ker \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$E_2 = \ker \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Now we verify that these vectors are eigenvectors for the eigenvalues 1, 2, and 3, respectively.

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$